Vectors According to Hostetler* evolution to 02/27/01 1:52 PM (last print/update)

10.1 Vectors in the Plane

*Using the notation of the text below with minor additions/changes. Calculus, 6th edition by Larson, Hostetler and Edwards; Houghton Mifflin Company

$\mathbf{v} = \overrightarrow{PQ}$ has ***initial point*** P and ***terminal point*** Q.

v is in ***standard position*** if it has its initial point at the origin.

$\mathbf{v} = \langle v_1, v_2 \rangle$ is the ***component form*** of **v**.

$\mathbf{v} = \langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle = \mathbf{v}$ is the ***component form*** of the vector $\mathbf{v} = \overrightarrow{PQ}$.

$\mathbf{u} = \mathbf{v}$ (**u** is ***equal or equivalent*** to **v**) if **u** and **v** have the same ***length*** (or ***magnitude***) and ***direction***. They do not have to be in the same position.

$\|\mathbf{v}\| = \|\overrightarrow{PQ}\|$ is the notation for the ***length*** (or ***magnitude***) of **v**. $\|\mathbf{v}\| = \|\langle v_1, v_2 \rangle\| = \sqrt{\mathbf{v} \bullet \mathbf{v}} = \sqrt{v_1^2 + v_2^2}$ (see dot product later)

$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$ ***is*** the ***length*** (or ***magnitude***) of $\mathbf{v} = \overrightarrow{PQ}$.

If $\|\mathbf{v}\| = 1$, then **v** is a ***unit vector***. $\|\mathbf{v}\| = 0$ if and only if **v** is the ***zero vector*** **0**.

Scalar Multiplication (scalar multiple): $k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$ (k is a number, **u** is a vector)

Note: *Scalar Multiplication* is not to be confused with the *Scalar Product* (Dot Product)

Vector Addition $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$

Negative $-\mathbf{u} = (-1)\langle u_1, u_2 \rangle = \langle -u_1, -u_2 \rangle$ ***Difference:*** $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle$

Properties of ***Vector Addition*** and ***Scalar Multiplication:***	
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ commutative	6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ distributive (sum of scalars over vector)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ associative	7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ distributive (scalar over sum of vectors)
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ identity for addition	8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$ identity & zero for multiplication
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ inverse for addition	9. $\|c\mathbf{u}\| = \|c\|\,\|\mathbf{u}\|$ the norm of the product of a scalar times a vector equals the product of the absolute value of the scalar times the norm of the vector
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$ associative for scalars times a vector	

The ***unit vector*** for given vector **v** is $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\mathbf{v}$ (Note: **u** is a scalar multiple of **v** and **u** has length 1)

Standard Unit Vectors $\mathbf{i} = \langle 1,0 \rangle$ *and* $\mathbf{j} = \langle 0,1 \rangle$

***Linear Combination of the Unit Vectors* i *and* j** for $\mathbf{u} = \overrightarrow{PQ}$

$\mathbf{u} = \langle q_1 - p_1, q_2 - p_2 \rangle = (q_1 - p_1)\mathbf{i} + (q_2 - p_2)\mathbf{j} = u_1\mathbf{i} + u_2\mathbf{j}$

Note: The magnitude of a vector represented by ***a scalar times a unit vector***, is the absolute value of the scalar. That is, $\|c\mathbf{u}\| = |c|$ when **u** is a unit vector.

Ex) $\|c\langle \cos a, \sin a \rangle\| = |c|$

The ***direction angle*** for $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is determined from $\tan q = b/a$ as $q = \tan^{-1}(b/a)$
(Note: The quadrant must be determined and theta adjusted accordingly.)

Triangle Inequality:
$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

The ***Component Form from magnitude and direction*** is: $\mathbf{v} = \|\mathbf{v}\|(\cos q)\mathbf{i} + \|\mathbf{v}\|(\sin q)\mathbf{j} = \langle v_1, v_2 \rangle$

About Vectors being Parallel - Slopes, Tangents, and Normals
Two vectors are **parallel** if they are nonzero scalar multiples of one another or, equivalently, if the line segments representing them are **parallel**. Similarly, a vector is **parallel** to a line if the segments that represent the vector are **parallel** to the line. The **slope** of a vector that is not **parallel** to the y-axis is the **slope** shared by the lines **parallel** to the vector. Thus, if a ≠ 0, the vector V = ai + bj has a well-defined **slope**, which can be calculated from the components of **v**.
Calculus: A vector is **tangent** or **normal** to a differentiable curve at a point if it is **parallel** or **normal** to the line that is **tangent** or **normal** to the curve at the point.

10.2 Space Coordinates and Vectors in Space

Vectors in the Space

Distance between two points $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Equation of a Sphere $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

Midpoint $(x, y, z) = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2} \right)$

Vectors in Space The definitions of Equality, Component Form, Length, Unit Vector, Vector Addition, and Scalar Multiplication are extensions of the two dimensional versions on the previous page

Parallel Vectors Two nonzero vectors **u** and **v** are parallel if there is some scalar *c* such that **u** = *c* **v**

The ***direction*** of **v** is given by $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ (unit vector)

The ***length and direction*** of **v** is given by $\|\mathbf{v}\| \frac{\mathbf{v}}{\|\mathbf{v}\|}$ (magnitude times its unit vector)

The vectors **A** and **B** have the ***same length and direction*** when $\frac{\mathbf{A}}{\|\mathbf{A}\|} = \frac{\mathbf{B}}{\|\mathbf{B}\|}$ or $\mathbf{A} = \|\mathbf{A}\| \frac{\mathbf{B}}{\|\mathbf{B}\|}$

The vectors **A** and **B** have ***opposite direction*** when $\frac{\mathbf{A}}{\|\mathbf{A}\|} = -\frac{\mathbf{B}}{\|\mathbf{B}\|}$ or $\mathbf{A} = -\|\mathbf{A}\| \frac{\mathbf{B}}{\|\mathbf{B}\|}$

The ***base vectors*** (in 3-space) are $\mathbf{i} = \langle 1,0,0 \rangle$, $\mathbf{j} = \langle 0,1,0 \rangle$ and $\mathbf{k} = \langle 0,0,1 \rangle$

The ***position vector*** **r** from the origin *O* to the typical point *P*(*x*, *y*, *z*) is $\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

10.3 The Dot Product of Two Vectors

Dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

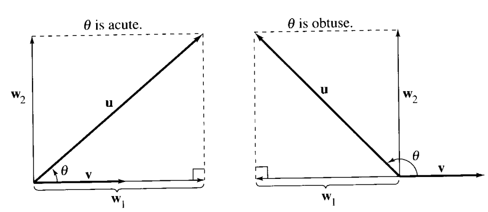
Dot product and Magnitude of $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2} \right)^2$ (see 4 below)

Properties of the ***Dot Product:***	
1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$ commutative	4. $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$ dot product equals the norm squared
2. $\mathbf{0} \bullet \mathbf{u} = 0$ zero factor	5. $c(\mathbf{u} \bullet \mathbf{v}) = c\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet c\mathbf{v}$ scalar commutes across dot product
3. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$ distributive (dot product over vector addition)	

Angle between two nonzero vectors in standard position $0 \le q \le p$, $\cos q = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$; $\cos q \|\mathbf{u}\| \, \|\mathbf{v}\| = \mathbf{u} \bullet \mathbf{v}$; $\cos q \|\mathbf{u}\| = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|}$

Vectors **u** and **v** are ***orthogonal (perpendicular)*** vectors if $\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$

Vector Components: Let **u** and **v** be nonzero vectors such that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\mathbf{w}_1$ is parallel to **v**. Then $\mathbf{w}_1$ and $\mathbf{w}_2$ are called ***vector components*** of **u**. The vector $\mathbf{w}_1$ is the ***projection*** of **u** onto **v** and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u}$ and $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

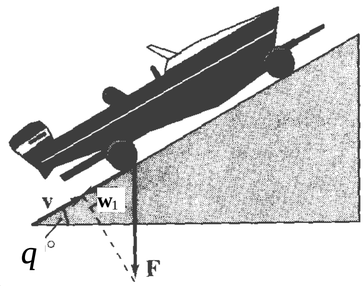


***Projection of* u *onto* v:** $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$ for $\mathbf{w}_1$ as above

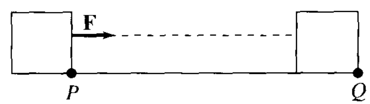
Direction Cosines $\cos\alpha = \dfrac{v_1}{\|\mathrm{v}\|}$, $\cos\beta = \dfrac{v_2}{\|\mathrm{v}\|}$, $\cos\gamma = \dfrac{v_3}{\|\mathrm{v}\|}$, (unit vector)

$\dfrac{\mathbf{v}}{\|\mathrm{v}\|} = \dfrac{v_1}{\|\mathrm{v}\|}\mathbf{i} + \dfrac{v_2}{\|\mathrm{v}\|}\mathbf{j} + \dfrac{v_3}{\|\mathrm{v}\|}\mathbf{k} = \cos\alpha\ \mathbf{i} + \cos\beta\ \mathbf{j} + \cos\gamma\ \mathbf{k}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Force: The projection of **F** (force) onto **v** is given by $\mathrm{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ where ***v is the unit vector*** along the ramp. The ***magnitude of $\mathbf{w}_1$ is the force needed*** to keep the boat on the ramp.



Note: The ***work*** *W* done by a ***constant force*** **F** ***acting along the line of motion*** of an object is given by *W* = (magnitude of force)(distance) = $\mathbf{F} \bullet \overrightarrow{PQ}$ (as at the right). If the ***constant force*** **F** ***is not acting along the line of motion***, the work *W* done by the force is $W = \left\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\right\| \left\|\overrightarrow{PQ}\right\| = (\cos\theta)\|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\| = \mathbf{F} \bullet \overrightarrow{PQ}$ (as below right). (Summarized below.)

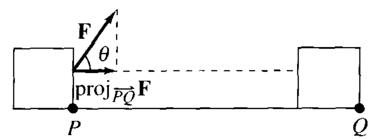


Work: The work done by a constant force **F** as its point of application moves along the vector *PQ* is given by any of the following:

1) $W = \left\|\text{proj}_{PQ}\mathbf{F}\right\| \left\|\overrightarrow{PQ}\right\|$ projection form

2) $W = \cos\theta \|\mathbf{F}\| \left\|\overrightarrow{PQ}\right\|$ angle form

3) $W = \mathbf{F} \bullet \overrightarrow{PQ}$ dot product form



10.4 The Cross Products of Two Vectors in Space

Vector product or ***cross product*** $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ Note: (2-dimensions) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix}$

Algebraic Properties of the Cross Product	
1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$	4. $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$	5. $\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\bullet\mathbf{w}$
3. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$	6. $c(\mathbf{u}\times\mathbf{v}) = (c\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(c\mathbf{v})$

Geometric Properties of the Cross Product	
1. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**	3. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ iff **u** and **v** are scalar multiples of each other
2. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ (magnitude of the torque vector)	4. $\|\mathbf{u}\times\mathbf{v}\| =$ area of a parallelogram with **u** and **v** adjacent
5. $\mathbf{u}\times\mathbf{v} = \pm(\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta)\mathbf{n}$ when **n** is a unit vector orthogonal to both **u** and **v** (torque vector)	

Triple Scalar or Box Product: $\left|(\mathbf{A}\times\mathbf{B})\bullet\mathbf{C}\right| = \left|\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C})\right| = \left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\|$

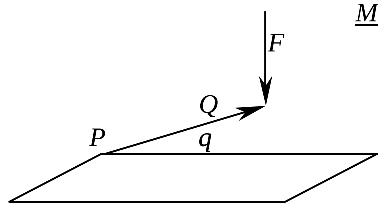
Area of a parallelepiped: $\left|\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C})\right| = \left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\|$ ← absolute value of the determinant (solid parallelogram)

$\mathbf{i}\times\mathbf{j} = -(\mathbf{j}\times\mathbf{i}) = \mathbf{k}$

$\mathbf{j}\times\mathbf{k} = -(\mathbf{k}\times\mathbf{j}) = \mathbf{i}$

$\mathbf{k}\times\mathbf{i} = -(\mathbf{i}\times\mathbf{k}) = \mathbf{j}$

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$



Moment of Force *F* about a point *P* is given by

$\mathbf{M} = \overrightarrow{PQ}\times\mathbf{F}$

10.5 Lines and Planes in Space

For $P(x_1, y_1, z_1)$, $Q(x, y, z)$, $R(x_2, y_2, z_2)$, and $S(x_3, y_3, z_3)$

Parametric equations of a Line in Space

A line *L* parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the parametric equations

$x = x_1 + at,\quad y = y_1 + bt,\quad z = z_1 + ct$.

$\overrightarrow{PQ} = t\langle a, b, c\rangle$

a, *b*, and *c* are called the direction numbers.

For non-zero *a*, *b*, and *c*, you can solve for *t* and eliminate it to obtain the

Symmetric Equations $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

Solve the parametric equations for *t*

Standard equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having a normal vector $\mathbf{n} = \langle a, b, c\rangle$ can be represented, in ***standard form***, by the equation

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The ***general form*** is $ax + by + cz + d = 0$

$\mathbf{n}\bullet\overrightarrow{PQ} = 0$

Given 3 points, $\mathbf{n} = \overrightarrow{PR}\times\overrightarrow{PS}$, then $\mathbf{n}\bullet\overrightarrow{PQ} = 0$

Normal to the plane Given the general equation of a plane $ax + by + cz + d = 0$, ***a normal to the plane*** is $\mathbf{n} = \langle a, b, c\rangle$

Angle between two planes $\cos q = \dfrac{\left|\mathbf{n}_1\bullet\mathbf{n}_2\right|}{\left\|\mathbf{n}_1\right\|\left\|\mathbf{n}_2\right\|}$

Consequently, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are

1) *Perpendicular* if $\mathbf{n}_1\bullet\mathbf{n}_2 = 0$

2) *Parallel* if $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$

Distance between a Plane and a Point Q (not in the plane) $D = \left\|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\right\| = \dfrac{\left|\overrightarrow{PQ}\bullet\mathbf{n}\right|}{\left\|\mathbf{n}\right\|}$ where *P* is a point in the plane and **n** is normal to the plane

The ***distance between the point*** $Q(x_0, y_0, z_0)$ ***(not in the plane) and the plane*** given by $ax + by + cz + d = 0$ is

$D = \dfrac{\left|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1) + d\right|}{\sqrt{a^2 + b^2 + c^2}}$ or $D = \dfrac{\left|ax_0 + by_0 + cz_0 + d\right|}{\sqrt{a^2 + b^2 + c^2}}$ where $P(x_1, y_1, z_1)$ is a point in the plane and

$d = -(ax_1 + by_1 + cz_1)$

Distance between a Point Q and a Line in Space $D = \dfrac{\left\| \overrightarrow{PQ} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|}$ where **u** is the direction vector for the line and *P* is a point on the line

10.6 Surfaces in Space

1. **Sphere** $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

2. **Plane** $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ The ***general form*** is $ax + by + cz + d = 0$

3. **Cylindrical Surfaces** $f(x) + g(y) = c$, $f(x) + h(z) = c$, and $g(y) + h(z) = c$
   Examples $z = y^2$, $x^2 + y^2 = 9$
   C is the generating curve, the parallel lines are the rulings, the cylinder is the set of all the parallel lines

4. **Quadratic Surfaces** $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$
   6 types, a trace is the intersection of a surface with a planes, traces are conics

5. **Surfaces of Revolution** $y = r(z)$ is the generating curve that is revolved about the *z*-axis to form a surface of revolution, the trace in $z = z_0$ is a circle with radius $y = r(z_0)$ and equation $x^2 + y^2 = [r(z_0)]^2$, the surface of revolution is $x^2 + y^2 = [r(z)]^2$
   a) Revolved about the *x*-axis $y^2 + z^2 = [r(x)]^2$, generating curve $y = f(x)$ or $z = f(x)$
   b) Revolved about the *y*-axis $x^2 + z^2 = [r(y)]^2$, generating curve $x = f(y)$ or $z = f(y)$
   c) Revolved about the *z*-axis $x^2 + y^2 = [r(z)]^2$, generating curve $x = f(z)$ or $y = f(z)$

10.7 Cylindrical and Spherical Coordinates

Cylindrical Coordinates $P = (r, q, z)$; r, q as r, q before (polar), *z* is the directed distance between (r, q) and *P*.

To ***change coordinates between rectangular and cylindrical*** in equations, use:

Cylindrical to Rectangular: $x = r\cos q$, $y = r\sin q$, $z = z$

Rectangular to cylindrical: $r^2 = x^2 + y^2$, $\tan q = \dfrac{y}{x}$, $z = z$

Spherical Coordinates $P = (r, q, f)$; r, q as r, q before, f is the angle between $\overrightarrow{OP}$ and the *z-axis.*

To ***change coordinates between rectangular and spherical*** in equations, use:

Spherical to Rectangular: $x = r\sin f \cos q$, $y = r\sin f \sin q$, $z = r\cos f$, $\sqrt{x^2 + y^2} = r\sin f$

Rectangular to Spherical: $r^2 = x^2 + y^2 + z^2$, $\tan q = \dfrac{y}{x}$, $f = \arccos\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

To ***change coordinates between cylindrical and spherical*** in equations, use:

Spherical to cylindrical ($r \geq 0$): $r^2 = r^2 \sin^2 f$, $q = q$, $z = r\cos f$

Cylindrical to Spherical ($r \geq 0$): $r = \sqrt{r^2 + z^2}$, $q = q$, $f = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$