

Vectors More Advanced

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The direction of \mathbf{v} is given by $\frac{\mathbf{v}}{\|\mathbf{v}\|}$.

Note: The magnitude
 $\|\mathbf{A}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = |\mathbf{A}|$
 (notation)

The length and direction of \mathbf{v} is given by $\|\mathbf{v}\| \frac{\mathbf{v}}{\|\mathbf{v}\|}$.

The vectors \mathbf{A} and \mathbf{B} have the same length and direction when $\frac{\mathbf{A}}{\|\mathbf{A}\|} = \frac{\mathbf{B}}{\|\mathbf{B}\|}$ or $\mathbf{A} = \|\mathbf{A}\| \frac{\mathbf{B}}{\|\mathbf{B}\|}$.

The vectors \mathbf{A} and \mathbf{B} have opposite direction when $\frac{\mathbf{A}}{\|\mathbf{A}\|} = -\frac{\mathbf{B}}{\|\mathbf{B}\|}$ or $\mathbf{A} = -\|\mathbf{A}\| \frac{\mathbf{B}}{\|\mathbf{B}\|}$.

The base vectors (in 3-space) are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

The position vector \mathbf{r} from the origin O to the typical point $P(x, y, z)$ is $\mathbf{r} = OP = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Note: See other vector pages for definitions of vector addition, subtraction, scalar multiplication, magnitude, and distance. The same rules apply in 3-space as those in 2-space when extended to three components.

Direction Cosines $\cos \mathbf{a} = \frac{v_1}{\|\mathbf{v}\|}$, $\cos \mathbf{b} = \frac{v_2}{\|\mathbf{v}\|}$, $\cos \mathbf{g} = \frac{v_3}{\|\mathbf{v}\|}$,

(unit vector) $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = \cos \mathbf{a} \mathbf{i} + \cos \mathbf{b} \mathbf{j} + \cos \mathbf{g} \mathbf{k}$, and $\cos^2 \mathbf{a} + \cos^2 \mathbf{b} + \cos^2 \mathbf{g} = 1$.

Projection of u onto v: $\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$ for \mathbf{w}_1 as above. (also see "Vectors" reference handout for Force and Work)

The Standard Equation for the Sphere of Radius a and center (x_0, y_0, z_0) is $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$.

The scalar product (or dot product) of \mathbf{A} and \mathbf{B} is $\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \cos \mathbf{q} = a_1 b_1 + a_2 b_2 + a_3 b_3$

where \mathbf{q} is the smaller angle made by \mathbf{A} and \mathbf{B} when their initial points coincide.

The angle between the two nonzero vectors \mathbf{A} and \mathbf{B} is $\mathbf{q} = \cos^{-1} \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\| \|\mathbf{B}\|} \right)$.

Properties of Vector Addition and Scalar Multiplication: These are the same as 2-space with a 3rd component added.

Vectors \mathbf{A} and \mathbf{B} are orthogonal when $\mathbf{A} \cdot \mathbf{B} = 0$.

The projection of \mathbf{A} onto \mathbf{B} is $\text{proj}_{\mathbf{A}} \mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A} = \frac{\mathbf{A} \cdot \mathbf{B}}{\|\mathbf{A}\|^2} \mathbf{A}$ (Note: $\mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\|^2$.)

The formula for writing B as a vector parallel to A plus a vector orthogonal to A. $\mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A} + \left(\mathbf{B} - \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A} \right)$

where $\mathbf{B}_1 = \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}$, $\mathbf{B}_2 = \mathbf{B} - \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \mathbf{A}$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$.

A line through $P(x_0, y_0)$ that is normal (perpendicular) to $N = A\mathbf{j} + B\mathbf{j}$ is given by $A(x - x_0) + B(y - y_0)$ or $Ax + By = C$, where $C = Ax_0 + By_0$.

About Vectors being Parallel - Slopes, Tangents, and Normals

Two vectors are **parallel** if they are nonzero scalar multiples of one another or, equivalently, if the line segments representing them are **parallel**. Similarly, a vector is **parallel** to a line if the segments that represent the vector are **parallel** to the line. The **slope** of a vector that is not **parallel** to the y-axis is the **slope** shared by the lines **parallel** to the vector. Thus, if $a \neq 0$, the vector $\mathbf{V} = a\mathbf{i} + b\mathbf{j}$ has a well-defined **slope**, which can be calculated from the components of \mathbf{v} .

Calculus: A vector is **tangent** or **normal** to a differentiable curve at a point if it is **parallel** or **normal** to the line that is **tangent** to the curve at the point.

Vector product or cross product: $\mathbf{A} \times \mathbf{B} = n \|\mathbf{A}\| \|\mathbf{B}\| \sin \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ Note: (2-dimensions) $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix}$

Because n is a unit vector, the magnitude of $\|\mathbf{A} \times \mathbf{B}\| = \|(n \|\mathbf{A}\| \|\mathbf{B}\| \sin \mathbf{q})\| = \|\mathbf{A}\| \|\mathbf{B}\| \sin \mathbf{q}$ which equals the area of a parallelogram.

Nonzero vectors \mathbf{A} and \mathbf{B} are parallel if and only if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ (Note: $\mathbf{A} \times \mathbf{A} = \mathbf{0}$)

The cross product is not commutative, but $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$

Scalar Distributive Law: $(s \mathbf{A}) \times \mathbf{B} = s(\mathbf{A} \times \mathbf{B})$

Vector Distributive Laws: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ and $(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = \mathbf{B} \times \mathbf{A} + \mathbf{C} \times \mathbf{A}$

Special case: $(-\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (-\mathbf{B}) = -(\mathbf{A} \times \mathbf{B})$

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= -(\mathbf{j} \times \mathbf{i}) = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= -(\mathbf{k} \times \mathbf{j}) = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= -(\mathbf{i} \times \mathbf{j}) = \mathbf{j} \\ \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{aligned}$$

Triple Scalar or Box Product: $|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}| = \|(\mathbf{A} \times \mathbf{B})\| \|\mathbf{C}\| \cos \mathbf{q} = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Area of a parallelepiped: $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (solid parallelepiped)

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Position vector with component functions $\mathbf{r} = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Tangent vector to the curve and velocity vector $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}$ (derivative of position)

Speed $Speed = \|\mathbf{v}\|$ (magnitude of velocity)

Acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$ (derivative of velocity)

Direction of motion $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ (direction of motion at time t)

Velocity $\|\mathbf{v}\| \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$ (as a product of its speed and direction)

Differentiation Rules for Vector Functions If \mathbf{r} , \mathbf{u} and \mathbf{v} are differential functions of t and c a differential function of s , then

Constant Function $\frac{d}{dt} \mathbf{C} = \mathbf{0}$

Scalar Multiple $\frac{d}{dt} (c\mathbf{u}) = c \frac{d\mathbf{u}}{dt}$

Sum $\frac{d}{dt} (\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$

Difference $\frac{d}{dt} (\mathbf{u} - \mathbf{v}) = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$

Dot Product $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt}$

Cross Product $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}$ (maintain order)

Chain Rule (short form) $\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \frac{dt}{ds}$

The derivative of a function of constant length is always orthogonal to the vector (direction change is at right angles). $\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0$

Indefinite integral of r with respect to t $\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}$, \mathbf{R} is any antiderivative of \mathbf{r} .

Definite integral of r from a to b $\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}$ for $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Ideal Projectile Motion

Motion when fired from the origin at time t = 0

Vector form: $\mathbf{r} = (v_0 \cos \mathbf{a})t \mathbf{i} + \left(h + (v_0 \sin \mathbf{a})t - \frac{1}{2}gt^2 \right) \mathbf{j}$

Cartesian equation form: $y = -\left(\frac{g}{2v_0^2 \cos^2 \mathbf{a}} \right) x^2 + (\tan \mathbf{a})x$

Parametric form: $x = (v_0 \cos \mathbf{a})t$, $y = (v_0 \sin \mathbf{a})t - \frac{1}{2}gt^2$

Firing from (x_0, y_0) : $x = x_0 + (v_0 \cos \mathbf{a})t$, $y = y_0 + (v_0 \sin \mathbf{a})t - \frac{1}{2}gt^2$

Height, Flight Time, and Range

Maximum Height: $y_{\max} = \frac{(v_0 \sin \mathbf{a})^2}{2g}$

Flight Time (time to impact): $t = \frac{2v_0 \sin \mathbf{a}}{g}$

Range (distance to point of impact): $R = \frac{v_0^2 \sin 2\mathbf{a}}{g}$

Length of a smooth curve $\mathbf{r} = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

Arc length parameter along a smooth curve from t_0 to t

$s(t) = \int_{t_0}^t \sqrt{[x'(I)]^2 + [y'(I)]^2 + [z'(I)]^2} dI = \int_{t_0}^t \|\mathbf{v}(I)\| dI$

(note: $\frac{ds}{dt} = \|\mathbf{v}\|$ and $\frac{dt}{ds} = \frac{1}{\|\mathbf{v}\|}$ and $\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{ds} = \mathbf{v} \left(\frac{1}{\|\mathbf{v}\|} \right) = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\left\| \frac{d\mathbf{r}}{ds} \right\| = \left\| \mathbf{v} \left(\frac{1}{\|\mathbf{v}\|} \right) \right\| = 1$ [length 1].)

Curvature, Torque and TNB Frames

Unit tangent vector $\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$

Principal unit normal vector $\mathbf{N} = \frac{1}{k} \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} / \left\| \frac{d\mathbf{T}}{dt} \right\| = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$

Binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Torsion

$\mathbf{t} = \frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} / dt}{\|\mathbf{v} \times \mathbf{a}\|^2}$

Curvature $\mathbf{k} = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$ (straight line $\mathbf{k} = \|\mathbf{0}\|$, circle $\mathbf{k} = 1/a$, $a = \text{radius}$) radius of curvature $\mathbf{r} = \frac{1}{\mathbf{k}}$

Tangential and normal components of acceleration $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ where $a_T = \frac{d}{dt} \|\mathbf{v}\|$ and $a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$

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Curvature, Torque and TNB Frames Summary formulas

Unit tangent vector $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$

Principal unit normal vector $\mathbf{N} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}$

Binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature $\mathbf{k} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$

Torsion

$$\mathbf{t} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\|\mathbf{v} \times \mathbf{a}\|^2}$$

Tangential and normal components of acceleration $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ where $a_T = \frac{d}{dt} \|\mathbf{v}\|$ and $a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$

Kepler's Laws (Planetary Motion and Satellites)

First (The Conic Section Law) - Eccentricity of the conic $e = \frac{r_0 v_0^2}{GM} - 1$, conic polar equation $r = \frac{(1+e)v_0}{1+e \cos q}$

Second (Equal Area Law) $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{dq}{dt} = \frac{1}{2} r_0 v_0$

Third (Time Distance Law) $\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$

G = universal gravitational constant ($6.6720 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$)
 M = the planet's mass (kilograms)
 t = time
 T = time to go around it's sun once
 A = area swept out by the radius vector \mathbf{r}
 v = velocity, v_0 = velocity at its perihelion position
 r = radius, r_0 = radius at its perihelion position
 q = counterclockwise angle from the perihelion position to \mathbf{r}
 a = semimajor axis...planets mean distance from its sun
 Force in newtons, distance in meters, time in seconds