Vectors in the Plane (elementary) evolution to 11/01/06 3:31 PM (last update)

$\mathbf{v} = \overrightarrow{PQ}$ has ***initial point*** *P* and ***terminal point*** *Q*.

v is in ***standard position*** if it has its initial point at the origin

$\mathbf{v} = \langle v_1, v_2 \rangle$ is the ***component form*** of **v**

$\mathbf{v} = \langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle = \mathbf{v}$ is the ***component form*** of the vector $\mathbf{v} = \overrightarrow{PQ}$

$\mathbf{u} = \mathbf{v}$ (**u** is ***equal or equivalent*** to **v**) if **u** and **v** have the same ***length*** (or ***magnitude***) and ***direction***. They do not have to be in the same position

$\|\mathbf{v}\| = \|\overrightarrow{PQ}\|$ is the notation for the ***length*** (or ***magnitude***) of **v**. $\|\mathbf{v}\| = \|\langle v_1, v_2 \rangle\| = \sqrt{\mathbf{v} \bullet \mathbf{v}} = \sqrt{v_1^2 + v_2^2}$ (see dot product next page)

$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$ ***is*** the ***length*** (or ***magnitude***) of $\mathbf{v} = \overrightarrow{PQ}$.

If $\|\mathbf{v}\| = 1$, then **v** is a ***unit vector***. $\|\mathbf{v}\| = 0$ if and only if **v** is the ***zero vector*** **0**

Scalar Multiplication (scalar multiple): $k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$ (*k* is a number, **u** is a vector)

Note: *Scalar Multiplication* is not to be confused with the *Scalar Product* (Dot Product)

Vector Addition $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$

Negative $-\mathbf{u} = (-1)\langle u_1, u_2 \rangle = \langle -u_1, -u_2 \rangle$ ***Difference:*** $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle$

Properties of ***Vector Addition*** and ***Scalar Multiplication:***	
1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ commutative	6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ distributive (sum of scalars over vector)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ associative	7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ distributive (scalar over sum of vectors)
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ identity for addition	8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$ identity & zero for multiplication
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ inverse for addition	9. $\|c\mathbf{u}\| = \lvert c \rvert \, \|\mathbf{u}\|$ the norm of the product of a scalar times a vector equals the product of the absolute value of the scalar times the norm of the vector. Also, $\|c\mathbf{u}\| = \lvert c \rvert$ when **u** is a unit vector. See box below.
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$ associative for scalars times a vector	

The ***unit vector*** for given vector **v** is $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\mathbf{v}$ (Note: **u** is a scalar multiple of **v** and **u** has length 1). $\mathbf{u} = \langle \cos\alpha, \sin\alpha \rangle$

Note on 9 above: $\|c\mathbf{u}\| = \lvert c \rvert$ when **u** is a unit vector.

That is, the magnitude of a vector represented by ***a scalar times a unit vector***, is the absolute value of the scalar. That is,

$$\|c\langle \cos\alpha, \sin\alpha \rangle\| = \|\langle c\cos\alpha, c\sin\alpha \rangle\| = \sqrt{(c\cos\alpha)^2 + (c\sin\alpha)^2}$$

$$= \sqrt{c^2\cos^2\alpha + c^2\sin^2\alpha} = \sqrt{c^2(\cos^2\alpha + \sin^2\alpha)} = \sqrt{c^2 \cdot 1} = \sqrt{c^2} = \lvert c \rvert$$

Standard Unit Vectors: $\mathbf{i} = \langle 1, 0 \rangle$ *and* $\mathbf{j} = \langle 0, 1 \rangle$

***Linear Combination of the Unit Vectors* i and j** for $\mathbf{u} = \overrightarrow{PQ}$

$$\mathbf{u} = \langle q_1 - p_1, q_2 - p_2 \rangle = (q_1 - p_1)\mathbf{i} + (q_2 - p_2)\mathbf{j} = u_1\mathbf{i} + u_2\mathbf{j}$$

The ***direction angle*** for $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is determined from $\tan\theta = b/a$ as $\theta = \tan^{-1}(b/a)$
(Note: The quadrant must be determined and theta adjusted accordingly.)

The ***Component Form from magnitude and direction*** is: $\mathbf{v} = \|\mathbf{v}\|(\cos\theta)\mathbf{i} + \|\mathbf{v}\|(\sin\theta)\mathbf{j} = \langle v_1, v_2 \rangle$

Alternate Form: $\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle$ where $\langle \cos\theta, \sin\theta \rangle$ is a unit vector and $\|\mathbf{v}\|$ is the magnitude.

About Vectors being Parallel - Slopes, Tangents, and Normals

Two vectors are **parallel** if they are nonzero scalar multiples of one another or, equivalently, if the line segments representing them are **parallel**. Similarly, a vector is **parallel** to a line if the segments that represent the vector are **parallel** to the line. The **slope** of a vector that is not **parallel** to the *y* -axis is the **slope** shared by the lines **parallel** to the vector. Thus, if $a \neq 0$, the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ has a well-defined **slope**, which can be calculated from the components of **v**.

Calculus: A vector is **tangent** or **normal** to a differentiable curve at a point if it is **parallel** or **normal** to the line that is **tangent** to the curve at the point.

Vectors in the Plane (part 2) evolution to 11/01/06 3:31 PM (last update)

Dot product of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2$

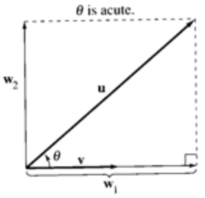
Dot product and Magnitude of $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2}\right)^2$ (see 4 below)

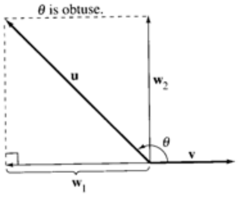
Properties of the ***Dot Product:***	
1. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$ commutative	4. $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$ dot product equals the norm squared
2. $\mathbf{0} \bullet \mathbf{u} = 0$ zero factor	5. $c(\mathbf{u} \bullet \mathbf{v}) = c\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet c\mathbf{v}$ scalar commutes across dot product
3. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$ distributive (dot product over vector addition)	

Angle between two nonzero vectors in standard position $0 \le \theta \le \pi$ $\cos\theta = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$; $\cos\theta \|\mathbf{u}\|\,\|\mathbf{v}\| = \mathbf{u} \bullet \mathbf{v}$; $\cos\theta \|\mathbf{u}\| = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|}$

Vectors **u** and **v** are ***orthogonal (perpendicular)*** vectors if $\mathbf{u} \bullet \mathbf{v} = \mathbf{0}$ and ***parallel*** if $\left|\dfrac{u_1}{u_2}\right| = \left|\dfrac{v_1}{v_2}\right|$.

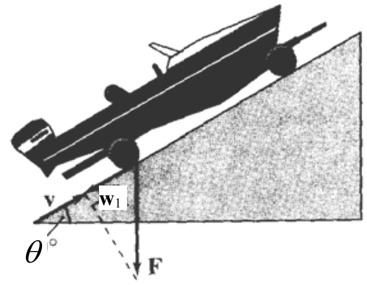
Vector Components Let **u** and **v** be nonzero vectors such that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\mathbf{w}_1$ is parallel to **v**. Then $\mathbf{w}_1$ and $\mathbf{w}_2$ are called ***vector components*** of **u**. The vector $\mathbf{w}_1$ is the ***projection*** of **u** onto **v** and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$



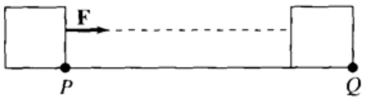


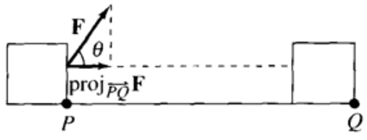
***Projection of* u *onto* v** $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$ for $\mathbf{w}_1$ as above

Force The projection of **u** = **F** (force) onto **v** is given by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ where ***v is the unit vector*** along the ramp. The ***magnitude of* w₁ *is the force needed*** to keep the boat on the ramp.



Note The ***work*** *W* done by a ***constant force*** **F** ***acting along the line of motion*** of an object is given by *W* = (magnitude of force)(distance) = $\mathbf{F} \bullet \overrightarrow{PQ}$ (as at the right). If the ***constant force*** **F** ***is not acting along the line of motion***, the work *W* done by the force is $W = \left\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\right\| \left\|\overrightarrow{PQ}\right\| = (\cos\theta)\|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\| = \mathbf{F} \bullet \overrightarrow{PQ}$ (as below right). (Summarized below.)





Work The work done by a constant force **F** as its point of application moves along the vector *PQ* is given by either of the following:

1) $W = \left\|\text{proj}_{PQ}\mathbf{F}\right\| \left\|\overrightarrow{PQ}\right\|$ projection form

2) $W = \cos\theta \|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\|$ angle form

3) $W = \mathbf{F} \bullet \overrightarrow{PQ}$ dot product form