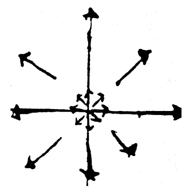
Vector Fields and Work

Def: A vector field is a function that assigns a vector to each point in the domain of a function. The general form of a vector field is

$$\mathbf{F} = M(x, y, z) + N(x, y, z) + P(x, y, z)$$

Examples of vector fields:

1) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$



2) The gradient of $f(x, y, z) = c$ is an example of a vector field.

$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ is a vector field that assigns a vector perpendicular to the surface $f(x, y, z) = c$, at every point (x, y, z).

3) Other examples: earth's gravitational field, electric force fields.

Work Done

To find the work done by $\mathbf{F} = M(x, y, z) + N(x, y, z) + P(x, y, z)$ in moving an object along a curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ from $t = a$ to $t = b$ is:

Work = (Net force acting on an object) x distance = $\int \mathbf{F} \bullet \mathbf{T}\,ds$ where $\mathbf{T}$ is the unit tangent vector of $\mathbf{r}(t)$.

Note that $\mathbf{F} \bullet \mathbf{T}$ represents, "the scalar component of $\mathbf{F}$ in the direction of $\mathbf{T}$". $\mathbf{F} \bullet \mathbf{T} = \|\mathbf{F}\|\|\mathbf{T}\|\cos q = \|\mathbf{F}\|\cos q$

In practice, to calculate work, we will need more practical formulas.

1. $\int \mathbf{F} \bullet \mathbf{T}\,ds = \int \mathbf{F} \bullet \mathbf{T}\|\mathbf{r}'(t)\|dt = \int \mathbf{F} \bullet \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\|dt = \int \mathbf{F} \bullet \mathbf{r}'(t)\,dt$

2. $= \int \mathbf{F} \bullet \frac{d\mathbf{r}}{dt}dt = \int \left(\frac{Mdx(t)}{dt} + \frac{Ndy(t)}{dt} + \frac{Pdz(t)}{dt} \right) dt = \int (Mdx + Ndy + Pdz)$

*If $\mathbf{F}$ represents a velocity field, then $\int \mathbf{F} \bullet \mathbf{T}\,ds$ is called the 'flow' or circulation and $\int \mathbf{F} \bullet \mathbf{N}\,ds$ is called 'flux'. Put another way, flux is the integral of the normal component of $\mathbf{F}$ and circulation is the integral of the tangential component of $\mathbf{F}$ over C.

Def: A vector field $\mathbf{F}$ is called conservative, if there is a differentiable function f, such that $\mathbf{F}$ is the gradient of f, i.e. $\mathbf{F} = \nabla f$ [f is called the potential function of $\mathbf{F}$]

Test for conservative fields in the plane:

$\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a conservative field if and only if $M_y = N_x$

Proof: If $\mathbf{F} = f_x\mathbf{i} + f_y\mathbf{j}$ then $M = f_x$ and $N = f_y$. In addition $M_y = f_{xy}$ *and* $N_x = f_{yx}$. Therefore, M_y and N_x are equal.

Test for conservative fields in space:

$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a conservative field if and only if the following conditions are satisfied:

$M_y = N_x$ $\quad\quad$ $M_z = P_x$ $\quad\quad$ $P_y = N_z$

Fundamental Theorem of Line Integrals:

Let C be a piecewise smooth curve, given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ where $a \le t \le b$. If $\mathbf{F}$ is a conservative field, then

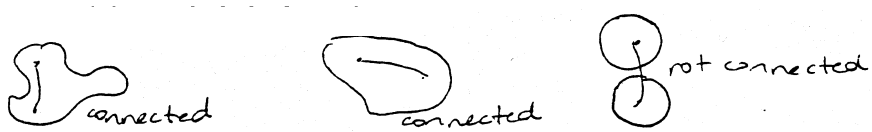
$$\int \mathbf{F} \bullet d\mathbf{r} = \int \nabla f \bullet d\mathbf{r} = f(B) - f(A) \text{ where } B = (x(b), y(b), z(b)) \text{ and } A = (x(a), y(a), z(a))$$

Proof:

$$\int \mathbf{F} \bullet d\mathbf{r} = \int \mathbf{F} \bullet (d\mathbf{r}/dt)dt$$

$$= \int \left(\frac{f_x dx}{dt} + \frac{f_y dy}{dt} + \frac{f_z dz}{dt} \right) dt = \int \frac{d}{dt}\left(f(x(t), y(t), z(t)) \right) dt = f((x(b), y(b), z(b)) - f((x(a), y(a), z(a))$$

Def: A region R is said to be connected, if any two points in the region can be joined by a smooth curve, lying entirely in the region.



Theorem: Let $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field defined over a connected region R and let C be a piecewise smooth curve in R. Then the following are equivalent:

a) $\mathbf{F}$ is conservative ($\mathbf{F} = \nabla f$ for some f)

b) $\int \mathbf{F} \bullet d\mathbf{r}$ is independent of path

c) $\int \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed curve in R