vector_fields.doc Tuesday, May 01, 2001 7:07 PM

Vector Field

For M and N functions of x and y, $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is a **vector field on a plane region *R***

For M, N and P functions of x, y, and z, $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a **vector field on a solid region *Q***

Inverse Square Field

For $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ the position vector, the field **F** is an **inverse square field** if

$\mathbf{F}(x,y,z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}$, for k a real number and $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$

Conservative Field

A field **F** is **conservative** and f is called the **potential function** for **F** if there exists f such that

$$\mathbf{F} = \nabla f$$

Test for Conservative Field in the Plane

The vector field given by $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

∇ is a **differential operator**

Curl of a Field

The **curl** of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\mathbf{curl}\,\mathbf{F}(x,y,z) = \nabla \times \mathbf{F}(x,y,z)$$

$$\mathbf{curl}\,\mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Test for Conservative Field in Space

The vector field given by $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\mathbf{curl}\,\mathbf{F}(x,y,z) = 0$$

Divergence of a Field

The **divergence** of $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is

$$\mathbf{div}\,\mathbf{F}(x,y) = \nabla \bullet \mathbf{F}(x,y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

The **divergence** of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\mathbf{div}\,\mathbf{F}(x,y,z) = \nabla \bullet \mathbf{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

If $\mathbf{div}\,\mathbf{F} = 0$, then **F** is said to be **divergence free**

Divergence and Curl

$\mathbf{div}\,(\mathbf{curl}\,\mathbf{F}) = 0$ for **F** a vector field

Inverse Square Fields

Gravitational Fields, Newton's Law (Central Force Field)

$$\mathbf{F}(x,y,z) = \frac{-Gm_1m_2}{x^2+y^2+z^2}\mathbf{u} = \frac{-Gm_1m_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

Force exerted on a particle of mass m_1 at (x, y, z) by a particle of mass m_2 at (0, 0, 0), unit vector **u** in direction from origin to (x, y, z), and G is the gravitational constant

Electric Force Fields, Coulomb's Law

$$\mathbf{F}(x,y,z) = \frac{-cq_1q_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

Force exerted on a particle with charge q_1 at (x, y, z) by a particle of charge q_2 at (0, 0, 0) where

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ and c depends on the choice of units

Line Integral of a Vector Field

$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet \mathbf{T}\,ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \bullet \mathbf{r}'(t)\,dt$$

where

$$\mathbf{F} = \mathbf{F}(x(t), y(t), z(t)),\ \mathbf{T} = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|,\ ds = \|\mathbf{r}'(t)\|dt$$

Differential Form (of above)

$$\int_C M\,dx + N\,dy + P\,dz$$

Fundamental Theorem of Line Integrals

Let C be a piecewise smooth curve, given by

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ where $a \le t \le b$. If **F** is a conservative field, then

$\int \mathbf{F} \bullet d\mathbf{r} = \int \nabla f \bullet d\mathbf{r} = f(B) - f(A)$ where

$B = (x(b), y(b), z(b))$ and $A = (x(a), y(a), z(a))$

Theorem: Let $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field defined over a connected region R and let C be a piecewise smooth curve in R. Then the following are equivalent:

a) **F** is conservative ($\mathbf{F} = \nabla f$ for some f)

b) $\int \mathbf{F} \bullet d\mathbf{r}$ is independent of path

c) $\int \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed curve in R

Line Integral of a Definite Integral

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

$$= \int_a^b f(x(t), y(t), z(t))\|\mathbf{r}'(t)\|dt$$

where f is a smooth region containing C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Green's Theorem

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (R always lies to the left). Then

$$\int M\,dx + N\,dy = \int_R\int \left(\frac{\partial N}{dx} - \frac{\partial N}{dx}\right) dA$$