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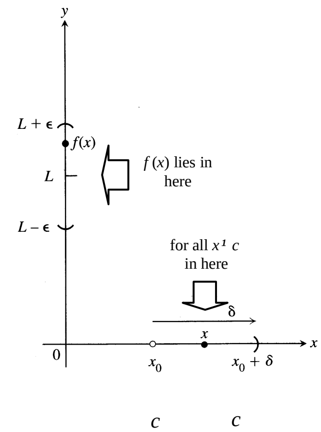
To prove that $\lim_{x\to 0}\frac{\sin x}{x}=1$, we will first examine the definition of a one-sided limit and prove the Sandwich Theorem.



DEFINITIONS

Right-hand limit: $\lim_{x\to c^+} f(x)=L$

The limit of *f* (*x*) as *x* approaches *c* from the right is the number *L* if the following criterion holds:

Given any *e* > 0 about *L* there exists a *d* > 0 to the right of *c* such that for all *x*

$c < x < c + d \quad \Rightarrow \quad | f(x) - L | < e.$

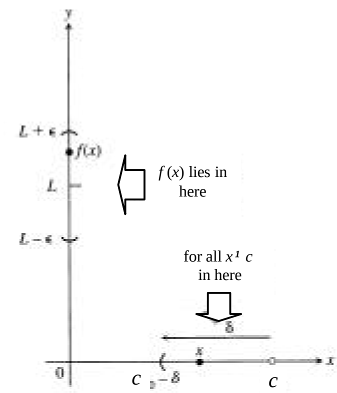
Left-hand limit: $\lim_{x\to c^-} f(x)=L$

The limit of *f* (*x*) as *x* approaches *c* from the left is the number *L* if the following criterion holds:

Given any *e* > 0 about *L* there exists a *d* > 0 to the left of *c* such that for all *x*

$c - d < x < c \quad \Rightarrow \quad | f(x) - L | < e.$

end



The Sandwich Theorem (Squeeze Theorem, Pinching Theorem)

Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about *c* and that $\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L$. Then $\lim_{x\to c} f(x)=L$.

Proof for Right-hand Limits.

Suppose $\lim_{x\to c^+} g(x)=\lim_{x\to c^+} h(x)=L$.

Then for any *e* > 0 there exists a *d* > 0 such that for all *x* the inequality $c < x < c + d$ implies

$L - e < g(x) < L + e$ and $L - e < h(x) < L + e.$

These inequalities combine with the inequality $g(x) \le f(x) \le h(x)$ to give

$L - e < g(x) \le f(x) \le h(x) < L + e,$

$L - e < f(x) < L + e,$ (10)

$- e < f(x) - L < e.$

Therefore, for all x, the inequality $c < x < c + d$ implies $| f(x) - L | < e$.

Proof for Left-hand Limits.

Suppose $\lim_{x\to c^-} g(x)=\lim_{x\to c^-} h(x)=L$.

Then for any *e* > 0 there exists a *d* > 0 such that for all *x* the inequality $c - d < x < c$ implies

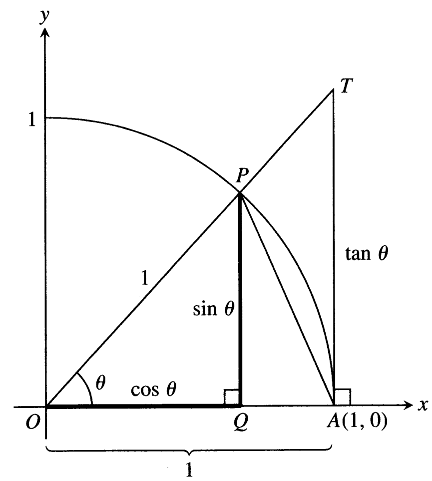
$L - e < g(x) < L + e$ and $L - e < h(x) < L + e$.

We conclude as before that for all *x* the inequality $c - d < x < c$ implies $| f(x) - L | < e$.

Theorem

If *q* is measured in radians, then $\lim_{q \to 0} \frac{\sin q}{q} = 1$ (5)

Proof



The figure for the proof of Theorem 6. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

Our plan is to establish Eq. (5) by showing that the right-hand and left-hand limits are both 1. We will then know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with values of *q* that are positive and less than *p* /2. When we compare the areas of Δ*OAP*, sector *OAP*, and Δ*OAT* in the figure, we see that

Area Δ*OAP* < area sector *OAP* < area Δ*OAT*. (6)

We can express these areas in terms of *q* as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta,$$

so that

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta. \qquad (10)$$

The inequality in (10) will go the same way if we divide all three terms by the positive number ½sin *q* :

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}. \qquad (11)$$

We next take reciprocals in (11), reversing the inequalities:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1. \qquad (12)$$

Because cos *q* approaches 1 as *q* approaches 0, the Sandwich Theorem tells us that

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1. \qquad (13)$$

The limit in Eq. (13) is a right-hand limit because we have been dealing with values of *q* between 0 and *p* /2. But, since the function sin *q /q* is even, we obtain the same limit as *q* approaches zero from the left. Therefore,

$$\lim_{q \to 0} \frac{\sin q}{q} = 1.$$

Other useful limits:

$$\lim_{q \to 0} x \sin\frac{1}{x} = 0 \qquad \lim_{q \to 0} \frac{1 - \cos q}{q} = 0 \qquad \lim_{q \to 0} \frac{\tan q}{q} = 1 \qquad \lim_{q \to 0} \sin\frac{1}{q} = undefined$$