Line Integrals

Line Integral of a Definite Integral

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

$$= \int_a^b f(x(t),y(t),z(t))\|\mathbf{r}'(t)\|dt$$

where f is a smooth region containing C given by
$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Line Integral of a Vector Field

$$W = \int_C \mathbf{F} \bullet dr = \int_C \mathbf{F} \bullet \mathbf{T}ds = \int_a^b \mathbf{F}(x(t),y(t),z(t)) \bullet \mathbf{r}'(t)dt$$

where

$$\mathbf{F} = \mathbf{F}(x(t),y(t),z(t)),\ \mathbf{T} = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|,\ ds = \|\mathbf{r}'(t)\|dt$$

Differential Form (of above)

$$\int_C Mdx + Ndy + Pdz$$

Fundamental Theorem of Line Integrals

Let C be a piecewise smooth curve, given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ where $a \le t \le b$. If $\mathbf{F}$ is a conservative field, then

$$\int \mathbf{F} \bullet d\mathbf{r} = \int \nabla f \bullet d\mathbf{r} = f(B) - f(A) \text{ where } B = (x(b),y(b),z(b)) \text{ and } A = (x(a),y(a),z(a))$$

Theorem: Let $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field defined over a connected region R and let C be a piecewise smooth curve in R.

Then the following are equivalent:

a) $\mathbf{F}$ is **conservative** ($\mathbf{F} = \nabla f$ for some f)

b) $\int \mathbf{F} \bullet d\mathbf{r}$ is **independent of path**

c) $\int \mathbf{F} \bullet d\mathbf{r} = 0$ for every **closed curve** in R

Green's Theorem

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (R always lies to the left). Then

$$\int Mdx + Ndy = \int_R \int \left(\frac{\partial N}{dx} - \frac{\partial N}{dx} \right) dA$$