Chain Rules for Several Variables

1. Partial $\frac{\partial z}{\partial x}$ treat as finding $\frac{dz}{dx}$ while treating all other variables as constants (except *x*)

2. Differential $dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$ when $w = f(x, y, z)$

3. When $w = u(x, y, z)$, $x = f(t)$, $y = g(t)$, $z = h(t)$, $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

4. When $w = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$, $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$

5. Implicit for $F(x,y) = 0$, $\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$ (*y* implicit),

6. Implicit for $F(x,y,z) = 0$, $\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$ and $\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$ (*z* implicit)

Directional Derivative of *f* in the direction of **u**

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos q + f_y(x, y)\sin q, \quad D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \bullet \mathbf{u}$$

($\mathbf{u} = \cos q\ \mathbf{i} + \sin q\ \mathbf{j}$, **u** a unit vector)

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z), \quad D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \bullet \mathbf{u}$$

($\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\mathbf{k}$, **u** a unit vector)

Gradient of *f*

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

$\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0) and $\nabla f(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0) (when $\nabla f(x_0, y_0) \neq 0$, $\nabla f(x_0, y_0, z_0) \neq 0$)

Properties of the Gradient of *f*

If $\nabla f(x, y) = 0$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all **u**

The direction of ***maximum*** increase of *f* is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is $\|\nabla f(x, y, z)\|$.

The direction of ***minimum*** increase of *f* is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is $-\|\nabla f(x, y, z)\|$.