Chain Rule

Suppose a function u is raised to a power. Then we would have u^n.

To find the derivative, we use the simple power rule on the u and then multiply the result by u'.

We then have $\dfrac{d(u^n)}{dx} = \dfrac{d(u^n)}{du} \cdot \dfrac{du}{dx} = nu^{n-1}\dfrac{du}{dx}$.   (1)

Letting $(u^n)' = d(u^n)/dx$ and $u' = du/dx$, we have $(u^n)' = nu^{n-1}u'$.

Or, in differential form and multiplying equation (1) by dx, $d(u^n) = nu^{n-1}du$.

We call this application of the chain rule, the General Power Rule. It also works in a similar way for other functions. For example, $\dfrac{d(e^u)}{dx} = \dfrac{d(e^u)}{du} \cdot \dfrac{du}{dx} = e^u\dfrac{du}{dx}$, or $(e^u)' = e^u u'$.

Example: Find y' when

$y = (3x+1)^2$.

We could avoid the chain rule and FOIL this to get

$y = 9x^2 + 6x + 1$.

Then

$y' = 18x + 6$.

Using the chain rule, we let $u = 3x + 1$. $u' = 3$.

Then

$y = (3x+1)^2$

becomes

$y = (u)^2 = u^2$

and

$y' = 2u \cdot u' = 2(3x+1)(3) = 6(3x+1) = 18x + 6$

as above.

Example: Find y' when

$y = (3x^2+1)^3$.

We could EXPAND this to get

$y = 27x^6 + 27x^4 + 9x^2 + 1$.

Then

$y' = 162x^5 + 108x^3 + 18x = 18x(9x^4 + 6x^2 + 1) = 18x(3x^2+1)^2$.

Using the chain rule, we let $u = 3x^2 + 1$. $u' = 6x$.

Then

$y = (3x^2+1)^3$

becomes

$y = (u)^3 = u^3$

and

$y' = 3u^2 \cdot u' = 3(3x^2+1)^2(6x) = 18x(3x^2+1)^2$

as above.

continued next page

Layered Chain Rule

Sometimes we have a chain rule within a chain rule within a chain rule and so on. The deeper they get gives us wonderful opportunities to train our minds in a problem solving technique that I call " component-izing" (Break it down into components).

Example: Find y' when

$$y = \left[(3x+1)^3 + 2\right]^2 .$$

We could EXPAND this to get

$$y = 729x^6 + 1458x^5 + 1215x^4 + 648x^3 + 243x^2 + 54x + 9 .$$

Then

$$y' = 4374x^5 + 7290x^4 + 4860x^3 + 1944x^2 + 486x + 54$$

and

$$y' = 54(3x+1)^2\left(9x^3 + 9x^2 + 3x + 1\right)$$

(after some fancy factoring).

Using the chain rule, we let $u = (3x + 1)^3 + 2$. $u' = 3v^2v' = 3(3x + 1)^2(3) = 9(3x + 1)^2$ where $v = 3x + 1$.

Then

$$y = \left[(3x+1)^3 + 2\right]^2$$

becomes

$$y = (u)^2 = u^2$$

and

$$y' = 2u \cdot u' = 2u(3v^2v') = 2\left[(3x+1)^3 + 2\right]\left[9(3x+1)^2\right]$$

$$y' = 18\left[(3x+1)^3 + 2\right](3x+1)^2 = 18\left(27x^3 + 27x^2 + 9x + 3\right)(3x+1)^2 = 54\left(9x^3 + 9x^2 + 3x + 1\right)(3x+1)^2$$

$$y' = 54(3x+1)^2\left(9x^3 + 9x^2 + 3x + 1\right)$$

as above.

With practice it becomes fairly easy.